Math 246B Lecture 7 Notes

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1 Averages of Subharmonic Functions

1.1 Convexity of averages of subharmonic functions

Last time, we proved the following theorem.

Theorem 1.1. *If $u \in SH(R_1 < |x| < R_2$, then $M(r) = \max_{|x|=r} u(x)$ is a convex function of $\log(r)$.*

This gave us a stronger form of the maximum principle. Here is a similar theorem.

Theorem 1.2. *Let $u \in SH(R_1 < |x| < R_2)$, let $0 \leq R_1 < R_2 \leq \infty$, and let*

$$I(r) = \frac{1}{2\pi r} \int_{|y|=r} u(y)\, ds(y). \qquad R_1 < r < R_2.$$

Then $I(r)$ is a convex function of $\log(r)$. If $u \in SH(|X| < R)$, then $I(r)$ is increasing, and $I(r) \xrightarrow{r \to 0^+} u(0)$.

Proof. Write

$$I(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it})\, dt.$$

Approximating u by a decreasing sequence of continuous functions, we see that $I(r)$ is upper semicontinuous. We claim that $I(r)$ satisfies the maximum principle: If $R_1 < r_1 < r_2 < R_2$, then

$$\max_{[r_1, r_2]} I(r) = \max(I(r_1), I(r_2)).$$

Let $R_1 < r_0 < R_2$, and let $\rho > 0$ be small. Let $|x| = r_0$, and write

$$\begin{aligned} u(x) &\leq \frac{1}{\pi \rho^2} \iint_{|y| \leq \rho} u(x+y)\, dy \\ &= \frac{1}{\pi \rho^2} \iint u(x+y) \mathbb{1}_{B_0(\rho)}(y)\, dy \end{aligned}$$

$$= \frac{1}{\pi\rho^2} \iint u(y) \mathbb{1}_{B_0(\rho)}(y-x)\, dy.$$

Integrating over $|x| = r_0$, we get

$$\begin{aligned} I(r) &\leq \frac{1}{2\pi r} \frac{1}{\pi \rho^2} \int_{|x|=r_0} \left[\iint u(y) \mathbb{1}_{B_0(\rho)}(y-x)\, dy \right] ds(x) \\ &= \frac{1}{2\pi r} \frac{1}{\pi \rho^2} \iint u(y) \left[\int_{|x|=r_0} \mathbb{1}_{B_0(\rho)}(y-x)\, ds(x) \right] dy \\ &= \frac{1}{2\pi r} \frac{1}{\pi \rho^2} \iint u(y) \psi(y)\, dy, \end{aligned}$$

where

$$\psi(y) = \int_{|x|=r_0} \mathbb{1}_{B_0(\rho)}(y-x)\, ds(x).$$

The function ψ gives us the 1-dimensional Lebesgue measure of the part of the circle $\{|z-x| = r_0\}$ contained in the ball $B(y, \rho)$. We have

- $\psi \geq 0$,
- ψ is continuous,
- $\psi(y) = \varphi(|y|)$ for some function φ.
- $\varphi(r) = 0$ for $|r - r_0| \geq \rho$
- $\varphi(r_0) > 0$.

We get

$$I(r) \leq \iint u(y)\varphi(|y|)\, dy = \iint_{\substack{0 \leq t \leq 2\pi \\ |r-r_0| \leq \rho}} u(re^{it})\varphi(r) r\, dr\, dt = \int \tilde{\varphi}(r) I(r)\, dr,$$

where $\tilde{\varphi}(r) = 2\pi r \varphi(r)$. So

$$I(r_0) \leq \int \tilde{\varphi}(r) I(r)\, dr.$$

If u is harmonic, then equality holds. In particular, using $u = 1$, we get

$$\int \tilde{\varphi}(r)\, dr = 1.$$

The sub-mean value inequality

$$I(r_0) \leq \int \tilde{\varphi}(r) I(r)\, dt$$

can now be used to prove the maximum principle for $I(r)$ in the usual way. This proves the claim.

To show that $I(r)$ is convex, let $R_1 < r_2 < r_2 < R_2$, and let $\tilde{}(r) = I(r) - a\log(r) - b$ be such that $\tilde{I}(r_j) \leq 0$ for $j = 1, 2$. We want to show that $\tilde{I}(r) \leq 0$ when $r_1 \leq r \leq r_2$. This follows from the maximum principle applied to the subharmonic function $u(x) = a\log|x| - b$.

Now assume that u subharmonic in $|x| < R$. We want to show that $I(r)$ is increasing in r. We have $I(r) = f(\log(r))$, where f is convex on $(-\infty, \log(R))$. We want to show that f is increasing, so it suffices to show that the right derivative $f'_{\text{right}} \geq 0$. If $f'_{\text{right}}(t_0) < 0$ for some t_0, write

$$f(t) \geq f(t_0) + f'_{\text{right}}(t_0)(t - t_0).$$

Letting $t \to -\infty$, we get that $f(t) \to +\infty$. So $I(r) \to +\infty$ as $r \to 0$. This is impossible, as u is locally bounded above.

Finally, we have for all small $r > 0$,

$$u(0) \leq I(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it})\, dt.$$

Using the upper semicontinuity of u at 0, we get that $I(r) \xrightarrow{r \to 0^+} u(0)$. □

Here is a special case of this theorem, applied to a harmonic function u.

Corollary 1.1. *Let u be harmonic in $R_1 < |x| < R_2$. Then*

$$I(r) = a\log(r) + b.$$

Proof. The theorem gives us that

$$\pm I(r) = \frac{1}{2\pi r} \int_{|x|=r} u(x)\, ds(x)$$

are convex functions of $\log(r)$. So $I(r)$ is an affine function of $\log(r)$. □

1.2 The Phragmén-Lindelöf principle

We would like to extend the maximum principle for subharmonic functions to unbounded domains.

Example 1.1. Let $\Omega = \{\mathrm{Im}(z) = x_2 > 0\}$, and let $i(x) = x_2$. This is harmonic, unbounded, and $u|_{\partial\Omega} = 0$. The idea is that we should be ok if we demand that the function does not grow too rapidly at ∞.

We will prove a general theorem which will allow us to do this. The original motivation of Phragmén and Lindelöf was the case of when Ω is a sector of the complex plane.